STIMULATING CRITICAL THINKING IN CALCULUS

Mark D. Turner
Cuesta College

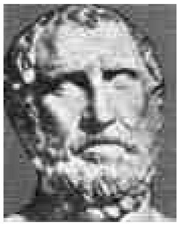
Copies of all handouts and overheads are available on my web site at:

http://academic.cuesta.edu/mturner/speak.htm

Calculus Message Board Assignment #1

All students are required to post a response on the message board before the deadline shown in the syllabus. Responses will be graded based on clarity, originality, depth, and for overall contribution to the discussion.

THE DICHOTOMY



Zeno of Elea was a pupil and friend of the philosopher Parmenides, and lived between 490 BC to about 425 BC. Zeno's philosophy, known as monism, centers around the theory that "all is one." That is, the many things that appear to exist individually are merely a single reality. Zeno proposed a number of paradoxes to support his argument against the idea that the world contains more than one thing. These paradoxes are derived from the assumption that if a magnitude can be divided then it can be divided infinitely often. One of these paradoxes, known as The Dichotomy, is described by Aristotle in his work *Physics*:

> "There is no motion because that which is moved must arrive at the middle of its course before it arrives at the end."

The idea behind the paradox is this: Suppose you would like to shoot an arrow at a target:

In order to reach its destination, the arrow must first travel half the distance to the target. Once there, half the distance still remains. From this position, the arrow must now travel half the remaining distance, and so on. Clearly, there is always some small amount of distance remaining between the tip of the arrow and the target. So, the arrow must always travel half that remaining distance first, thereby leaving the other half still to go. This argument compels us to conclude that the arrow cannot EVER reach the target. Of course, experience tells us that the arrow DOES reach the target! (All who disagree are welcome to stand in front of the target.) Hence we arrive at the paradox.

Let's look at the situation again from a numerical perspective. First the arrow travels half the distance to the target. Then it travels half the remaining distance (half of one-half), for a total of $1/2 + 1/4 = 3/4$ of the distance. Then it travels half the remaining distance again (half of one-fourth), for a total of $1/2 + 1/4 + 1/8 = 7/8$ of the distance. Repeating the process once more, the arrow travels half the remaining 1/8 of the distance for a total of $1/2 + 1/4 + 1/8 + 1/16 = 15/16$ of the distance. The arrow is certainly getting closer, but no matter how long we continue the process the arrow will NEVER reach the target because a little bit of distance ALWAYS remains after each step. So, how do we make sense of this dilemma?

Calculus Message Board Assignment #2

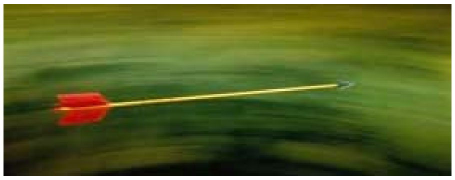
All students are required to post a response on the message board before the deadline shown in the syllabus. Responses will be graded based on clarity, originality, depth, and for overall contribution to the discussion.

THE ARROW

We look once more to Zeno for the inspiration of our second dilemma, which again concerns motion. Another of his paradoxes, known as The Arrow, is summarized by Aristotle:

> "If everything is either at rest or moving when it occupies a space equal to itself, while the object moved is in the instant, the moving arrow is unmoved."

This time the paradox focuses not on the divisibility of distance, but of time. Again, we shoot an arrow at a target. Consider any point in time during the arrow's flight. If there is such a thing as an "instant" in time, that is, a smallest interval of time that cannot be further divided, then the arrow cannot move during this instant. If the arrow did move and thereby cover some small distance, then there would have to be an even smaller instant of time (for example the time required for the arrow to move only half this distance). So the arrow must be at rest during each moment of its flight, yet still manages to get from one end of its journey to the other! (Imagine taking a picture of the arrow mid-flight; wouldn't it appear to be unmoving, as if suspended magically in air?)



Here is one writer's attempt to explain the paradox: "The human mind, when trying to give itself an accurate account of motion, finds itself confronted with two aspects of the phenomenon. Both are inevitable but at the same time they are mutually exclusive. Either we look at the continuous flow of motion; then it will be impossible for us to think of the object in any particular position. Or we think of the object as occupying any of the positions through which its course is leading it; and while fixing our thought on that particular position we cannot help fixing the object itself and putting it at rest for one short instant." (H Frankel, *Amer. J. Philology*, 1942)

So then how do we define the motion of an object at an instant in time? For example, we might refer to an automobile accident and say; "the vehicle was traveling at 65 miles per hour at the moment of impact." What does this mean? If we try to compute the velocity at any moment as the distance traveled divided by the total time, then we get 0/0, which is meaningless. Just how do we define velocity at a "moment" of time?

SCIENCE NEWS
COSMOLOGY

Breaking Down Time
Struggling scientists try to apply quantum theory to time, but time refuses to be quantized.
by Pamela L. Gay


Astronomers used Hubble Space Telescope images to investigate the nature of time
NASA

Quantum mechanics was born out of a mathematical leap of faith that wasn't initially thought to describe reality. Now, this still philosophically (and mathematically) difficult theory has grown to encompass most of physics. It successfully defines how large-scale physics breaks down at small scales for energy, momentum, position, and possibly time. It also specifies minimum measurable amounts (quanta) for these phenomena. However, observations by the Hubble Space Telescope have shaken the idea that time is quantized.

The quest of the last century has been to find a theory unifying general relativity with quantum mechanics. The Holy Grail of that quest is a theory of quantum gravity. While no complete theory has been established, scientists had some ideas of what it should include — a particle that carries the force of gravity and a quantum of time, for example. The gravity-carrying particle, dubbed a graviton by theorists, has too high an energy to be detected in current experiments. This leaves quantized time as the only testable aspect of quantum gravity.

If time is quantized like the energy of photons, there should be a certain minimum measurable quanta of time, called a "Planck time" (named after German physicist Max Planck, who originated the idea of quanta). This also means that life, like a movie, may appear to be a continuum of unfolding events but is really nothing more then a series of snapshots that together define the past and future.

Calculus Message Board Assignment #3

All students are required to post a response on the message board before the deadline shown in the syllabus. Responses will be graded based on clarity, originality, depth, and for overall contribution to the discussion.

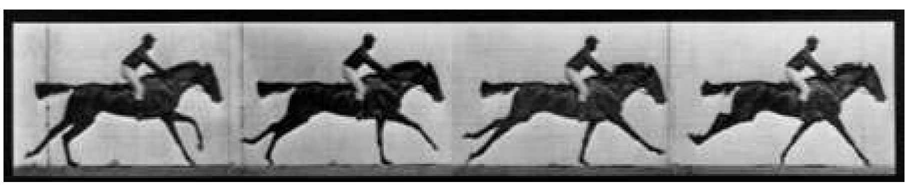
A 14th CENTURY MYSTERY

We have considered several perplexing issues related to motion that were first raised by Zeno. Continuing with the theme of motion, we will now direct our attention to one final mystery that was investigated in the 14th century:

If a body moves with varying speed, how far will it move in a given time?

If the object moves with CONSTANT speed, the answer is easy. Simply multiply the speed of the object by the length of time the object was in motion. Voila! However, if the velocity is changing over the time period, then this formula is no longer applicable.

Nicole d'Oresme, a French bishop, suggested an approach in which we subdivide the given interval of time into very small “instants”. (Is that Zeno I hear turning in his grave?) For example, the series of photos below captures the motion of a running horse at several such instants. If we assume that the speed cannot change in such an “instant” of time, then we can consider the speed to be a constant quantity during that instant.

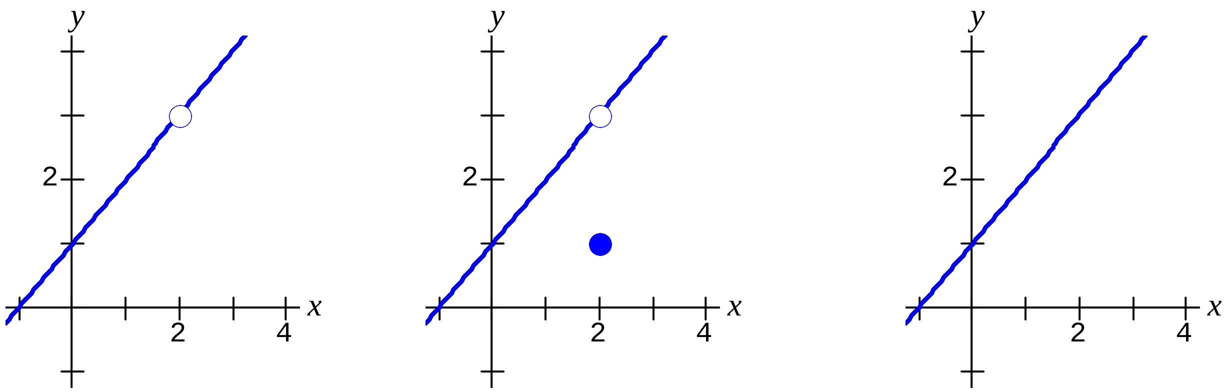


What was the perceptive bishop getting at? How does this help us find the distance traveled by the object?

Calculus: Limits I Name ___________________________

1. Consider the following three functions:

(a) $f(x)=\dfrac{x^2-x-2}{x-2}$ (b) $g(x)=\begin{cases} x+1 & x\neq 2 \\ 1 & x=2 \end{cases}$ (c) $h(x)=x+1$



How do $\lim\limits_{x\to 2} f(x)$, $\lim\limits_{x\to 2} g(x)$, and $\lim\limits_{x\to 2} h(x)$ compare? Write a paragraph to explain.

Calculus: Limits II Name ___________________________

Let $f(x) = \dfrac{\sqrt{x^2+9}-3}{x^2}$ and consider $\lim_{x \to 0} f(x)$.

(a) Is $f(x)$ defined at $x = 0$? _____________

(b) Investigate the limit numerically using the tables below.

x	$f(x)$
–0.1	
–0.001	
–0.00001	
–0.0000001	

x	$f(x)$
0.1	
0.001	
0.00001	
0.0000001	

Numerically, what do the tables suggest about this limit?

(c) Investigate the limit graphically. Graph $f(x)$ over the interval [–0.001, 0.001], then again over [–0.0001, 0.0001], and finally [–0.00001, 0.00001]. What does the graph suggest about this limit?

(d) Investigate this limit algebraically. Be sure to use correct notation in each step.

(e) Based on these three perspectives, what IS this limit? Can you explain any inconsistencies you observed between the three approaches?

Calculus: Limits II Name ___________________________

Let $f(x)=\dfrac{\sqrt{x^2+9}-3}{x^2}$ and consider $\lim\limits_{x\to 0} f(x)$.

(a) Is $f(x)$ defined at $x = 0$? ______________

(b) Investigate the limit numerically using the tables below.

x	***f*(*x*)**
–0.1	**0.16662040**
–0.001	**0.1666667**
–0.00001	**0.167**
–0.0000001	**0**

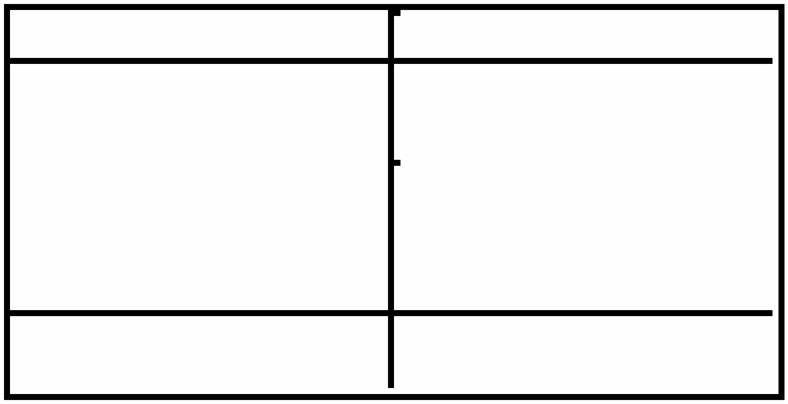
x	***f*(*x*)**
0.1	**0.16662040**
0.001	**0.1666667**
0.00001	**0.167**
0.0000001	**0**

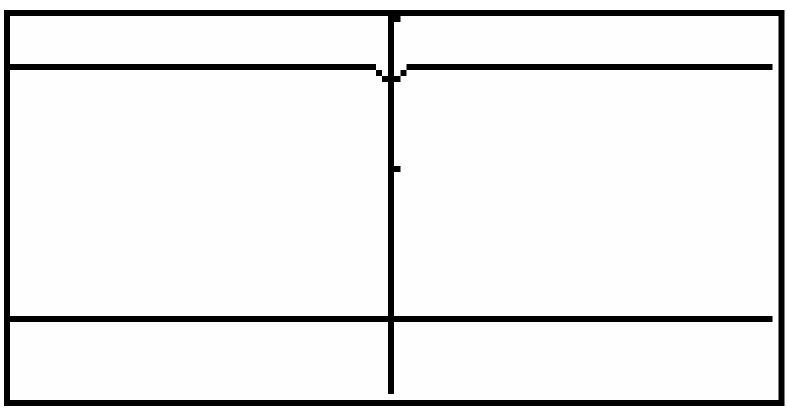
Numerically, what do the tables suggest about this limit?

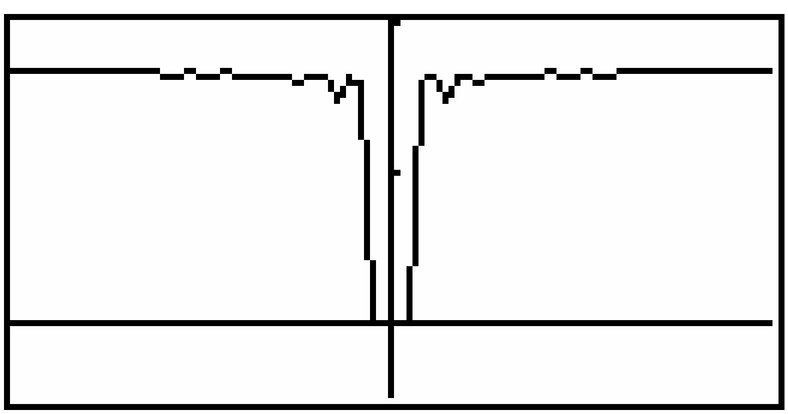
(c) Investigate the limit graphically. Graph $f(x)$ over the interval [–0.001, 0.001], then again over [–0.0001, 0.0001], and finally [–0.00001, 0.00001]. What does the graph suggest about this limit?

(d) Investigate this limit algebraically. Be sure to use correct notation in each step.

(e) Based on these three perspectives, what IS this limit? Can you explain any inconsistencies you observed between the three approaches?







Algebraically:

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$$

$$=\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$$

$$=\lim_{x\to 0}\frac{x^2+9-9}{x^2\left(\sqrt{x^2+9}+3\right)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3}$$

$$=\frac{1}{\sqrt{0+9}+3}$$

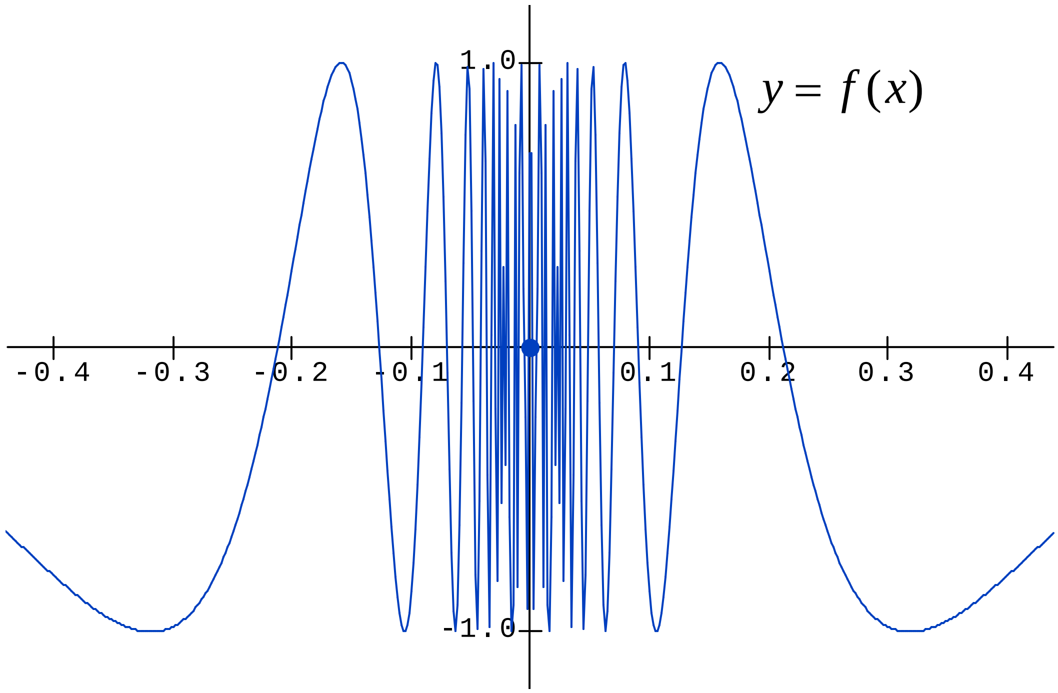
$$=\frac{1}{6}$$

Calculus: Continuity Name ___________________________

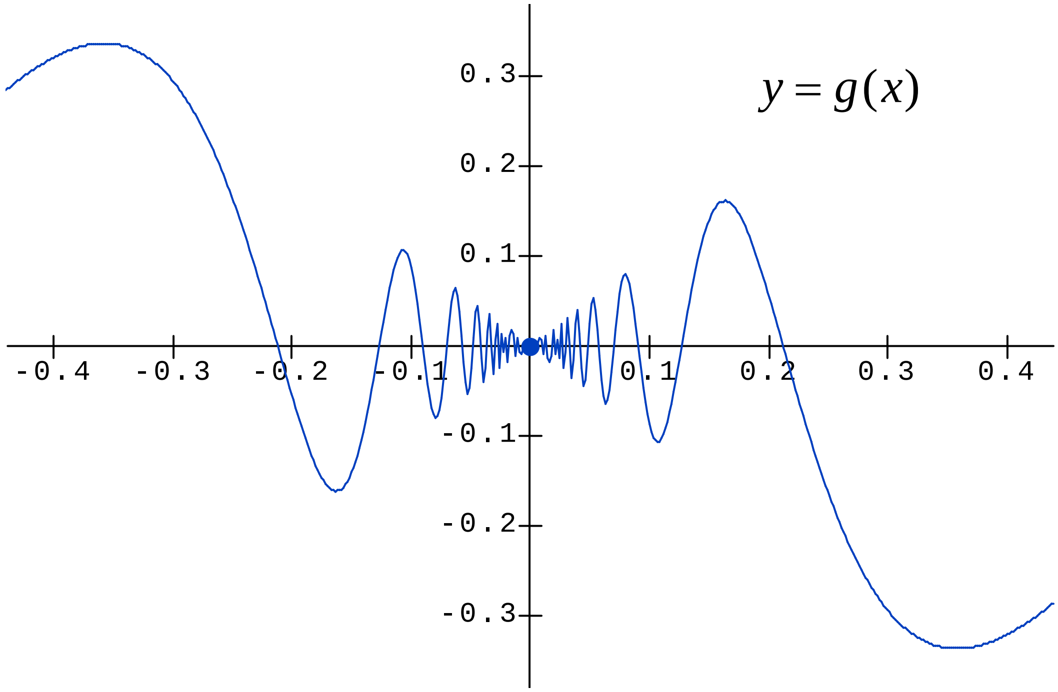
1. Suppose that $\lim_{x \to 4} r(x) = 13$. Which, if any, of the following statements *must* be true? Write a paragraph explaining your reasoning.
   (a) $r(x)$ is continuous at $x = 4$
   (b) $r(x)$ is defined at $x = 4$
   (c) $r(4) = 13$

2. Let $f(x) = \begin{cases} \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and consider whether *f* is continuous at $x = 0$. Make sure your calculator is in radian mode before continuing!
   (a) Is $f(x)$ defined at $x = 0$? __________ If so, what is $f(0)$? _________
   (b) Investigate the behavior of the function numerically and graphically near $x = 0$. Write down some observations of how *f* behaves as *x* approaches 0.
   (c) Is $f(x)$ continuous at $x = 0$? Explain why or why not. If not, determine if the discontinuity is removable or nonremovable.

3. Let $g(x) = \begin{cases} x\cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and consider whether *g* is continuous at $x = 0$. Make sure your calculator is in radian mode before continuing!
   (a) Is $g(x)$ defined at $x = 0$? __________ If so, what is $g(0)$? _________
   (b) Investigate the behavior of the function numerically and graphically near $x = 0$. Write down some observations of how *g* behaves as *x* approaches 0.
   (c) Is $g(x)$ continuous at $x = 0$? Explain why or why not. If not, determine if the discontinuity is removable or nonremovable.



1.0
-1.0
-0.4
-0.3
-0.2
-0.1
0.1
0.2
0.3
0.4
$y = f(x)$



0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.4
-0.3
-0.2
-0.1
0.1
0.2
0.3
0.4
$y = g(x)$

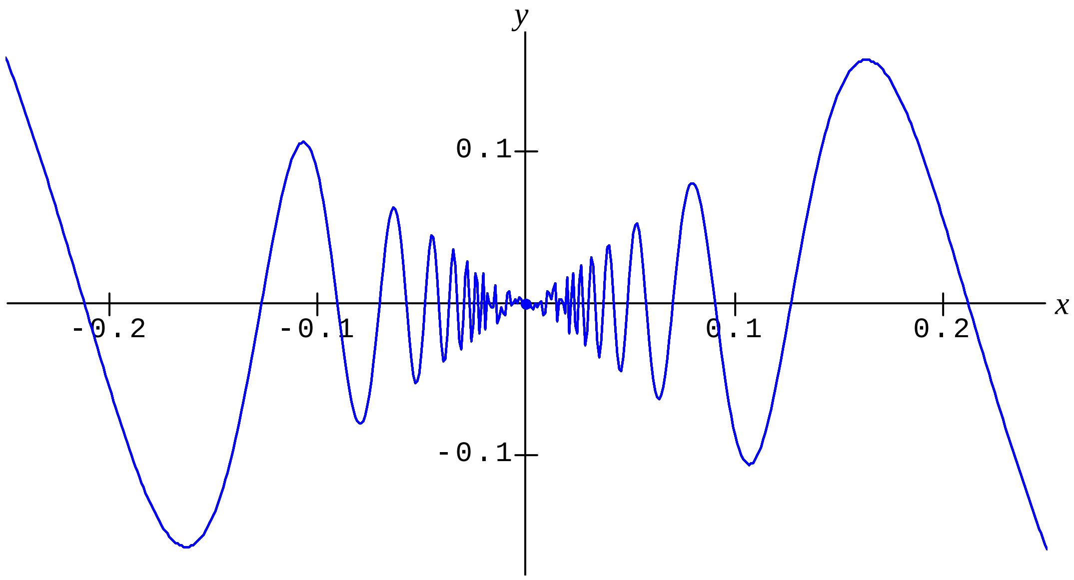
Calculus: Differentiability Name ____________________________

1. Consider the function $f(x)=\begin{cases} x\cos(1/x) & x\neq 0 \\ 0 & x=0 \end{cases}$. Previously we determined that this function is continuous at $x = 0$. Now you will investigate whether this function is *differentiable* at $x = 0$.

   (a) Use the limit definition of the derivative to determine algebraically whether $f'(0)$ exists. If so, determine its value; if not, explain why the limit fails to exist.

   (b) Using the graph of $f(x)$ provided below, explain why f is or is not differentiable at $x = 0$ from a *graphical* perspective.

   Hint: Recall that the derivative is the limit of the average rate of change between two points. Let one of the points be fixed at x = 0, and consider what happens to the average rate of change as the second point approaches this fixed point. Make some drawings!



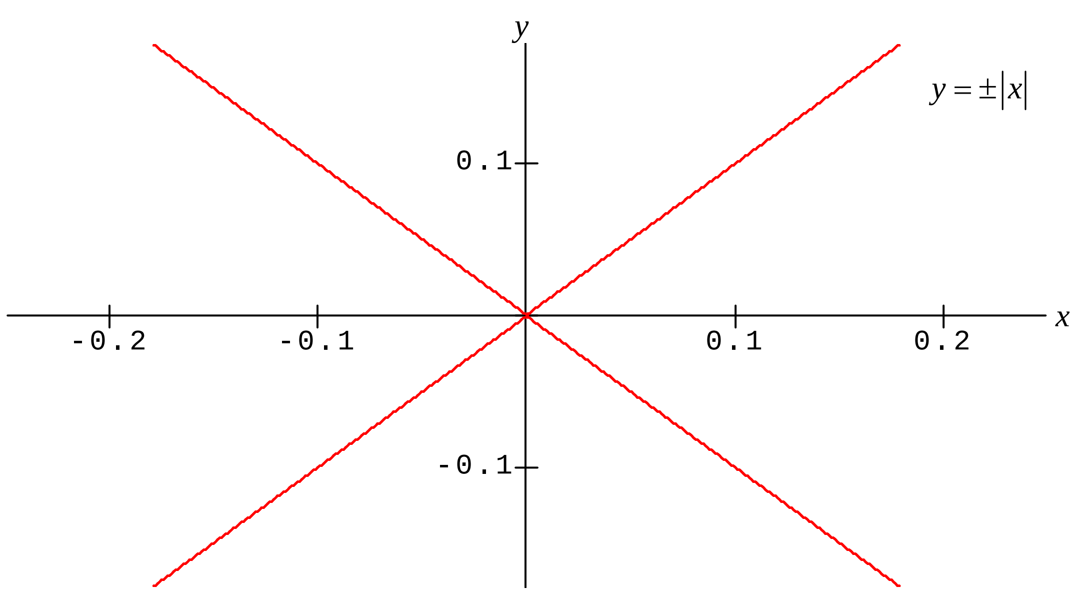
Explanation:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

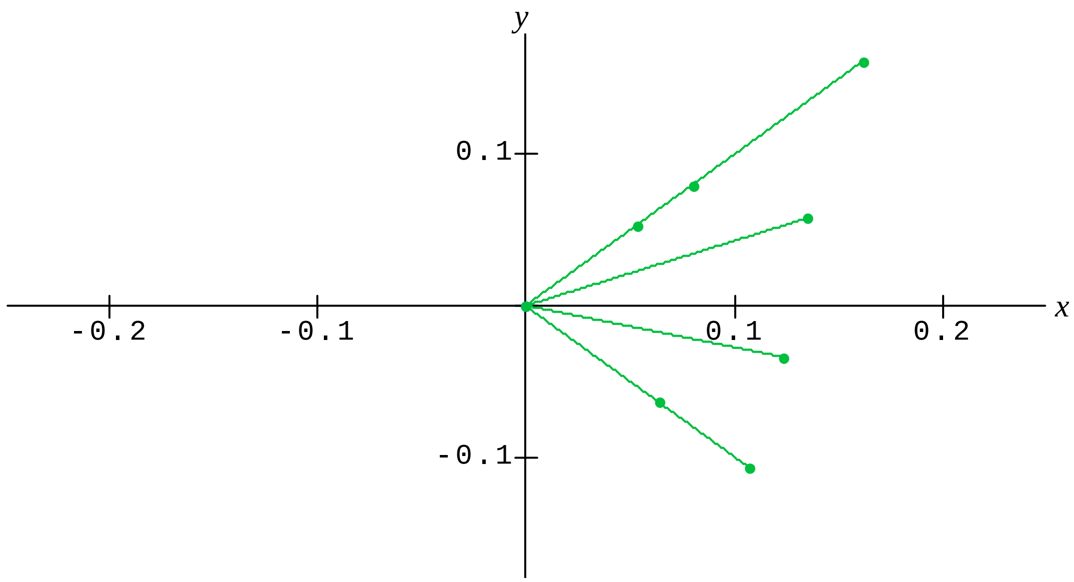
$$= \lim_{h\to 0} \frac{h\cos\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h\to 0} \cos\left(\frac{1}{h}\right)$$

This limit does not exist.



y
x
y = ±|x|
0.1
-0.1
-0.2
-0.1
0.1
0.2

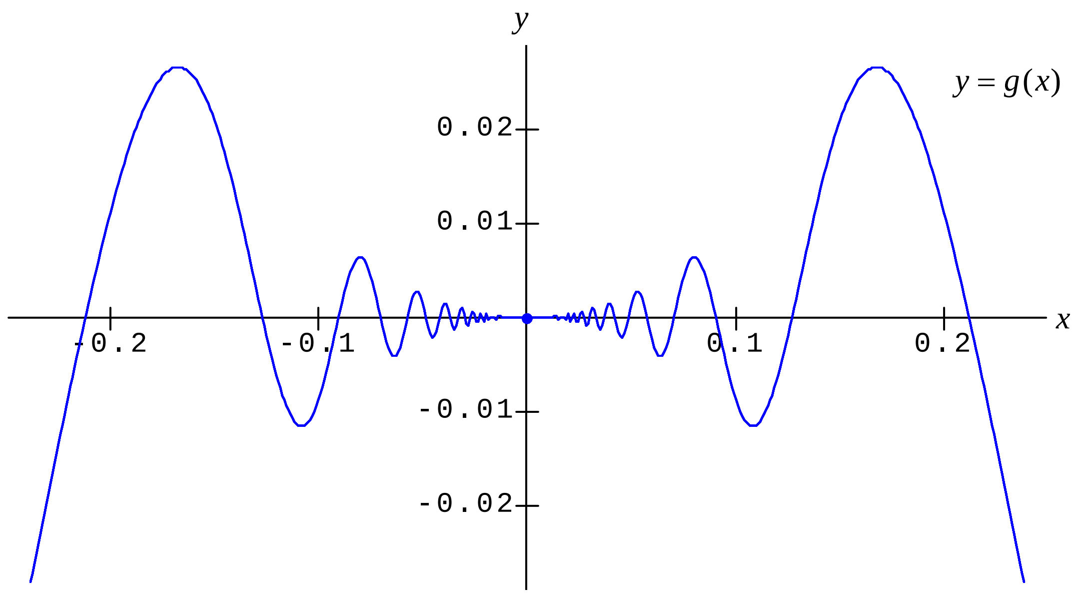

y
0.1
-0.1
-0.2
-0.1
0.1
0.2
x

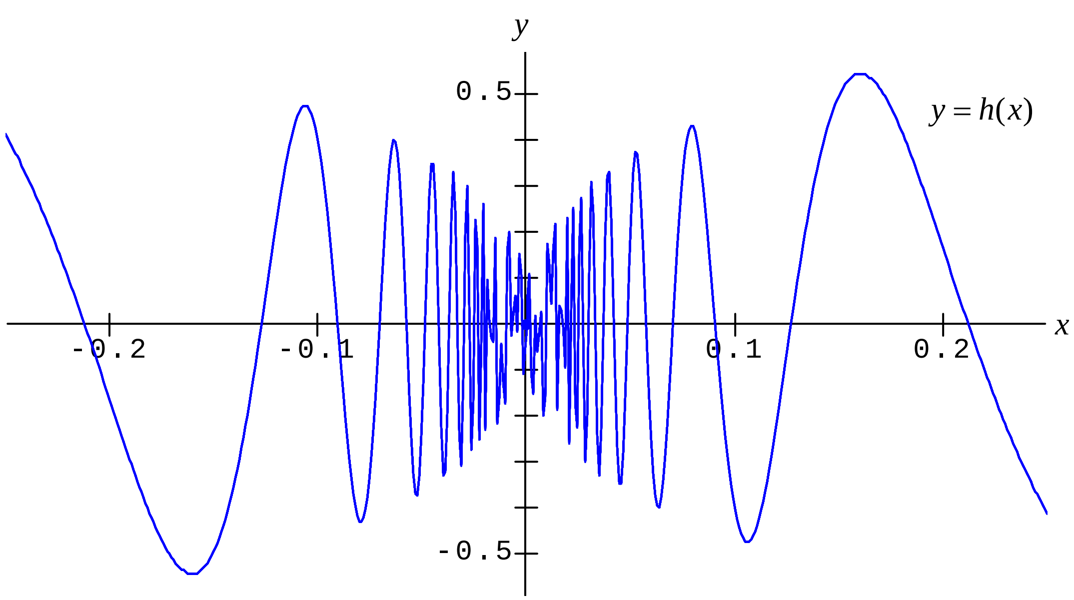
2. Now investigate the functions $g(x)=\begin{cases} x^2\cos(1/x) & x\neq 0 \\ 0 & x=0 \end{cases}$ and $h(x)=\begin{cases} x^{1/3}\cos(1/x) & x\neq 0 \\ 0 & x=0 \end{cases}$.

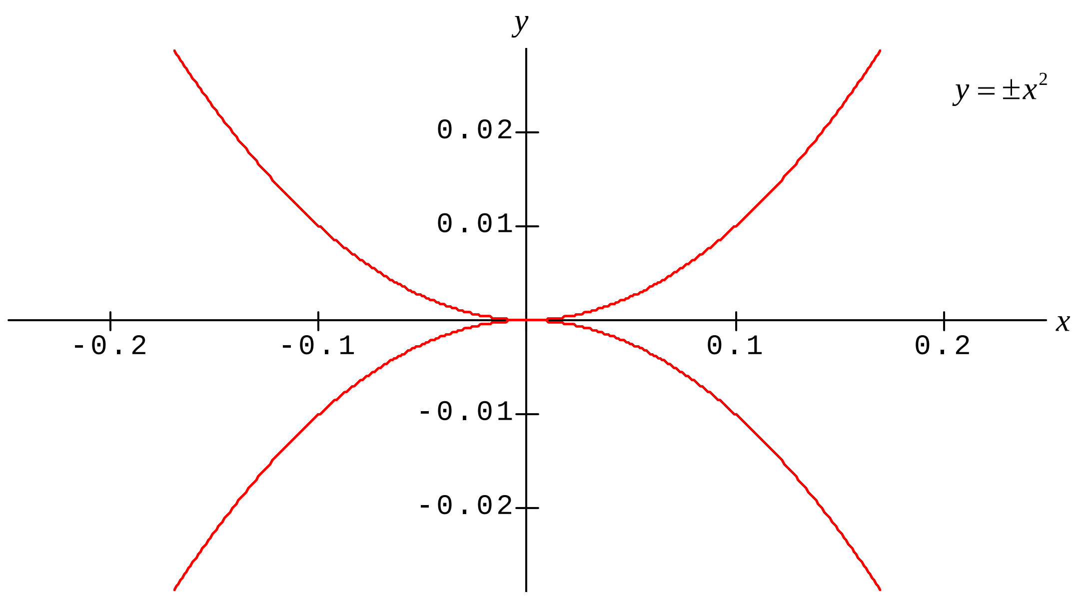
(a) Use the Squeeze Theorem to show algebraically that both functions are continuous at $x = 0$. (Attach your work on separate paper.)

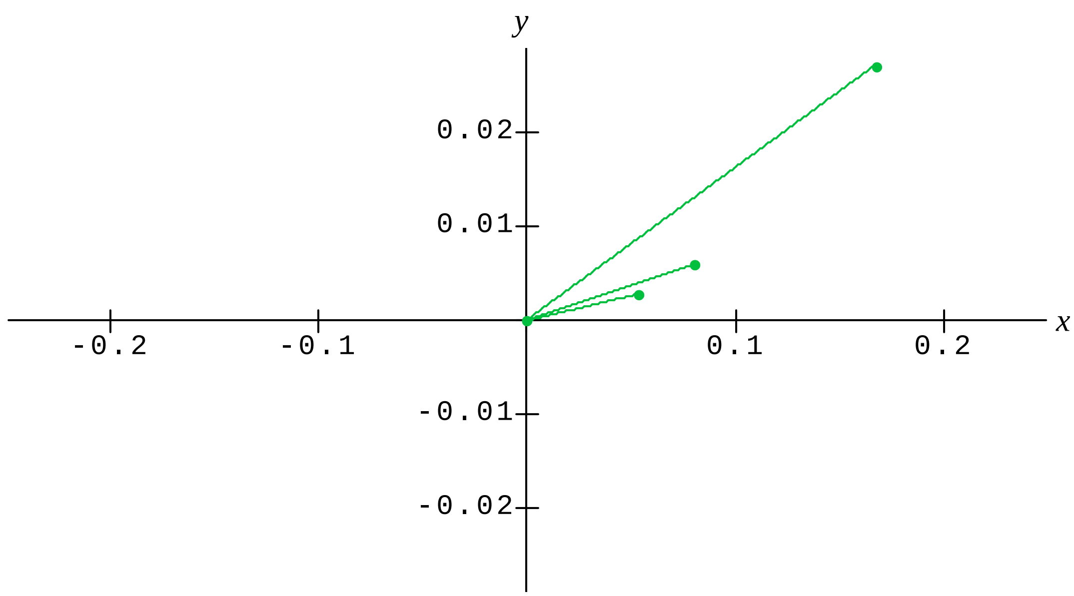
(b) Which of these functions, if any, are differentiable at $x = 0$? Justify your answer algebraically using the limit definition of the derivative. (Attach your work on separate paper.) *Hint: You may need to use the Squeeze Theorem again.)*

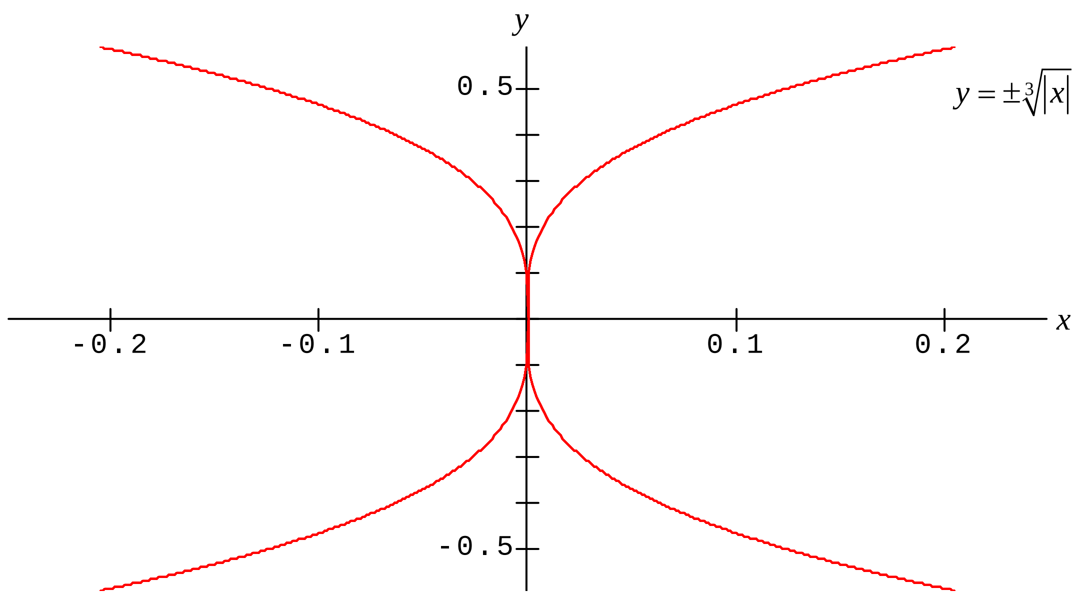
(c) Use the graphs of $g(x)$ and $h(x)$ provided below to justify your conclusions in part (b) from a *graphical* perspective.

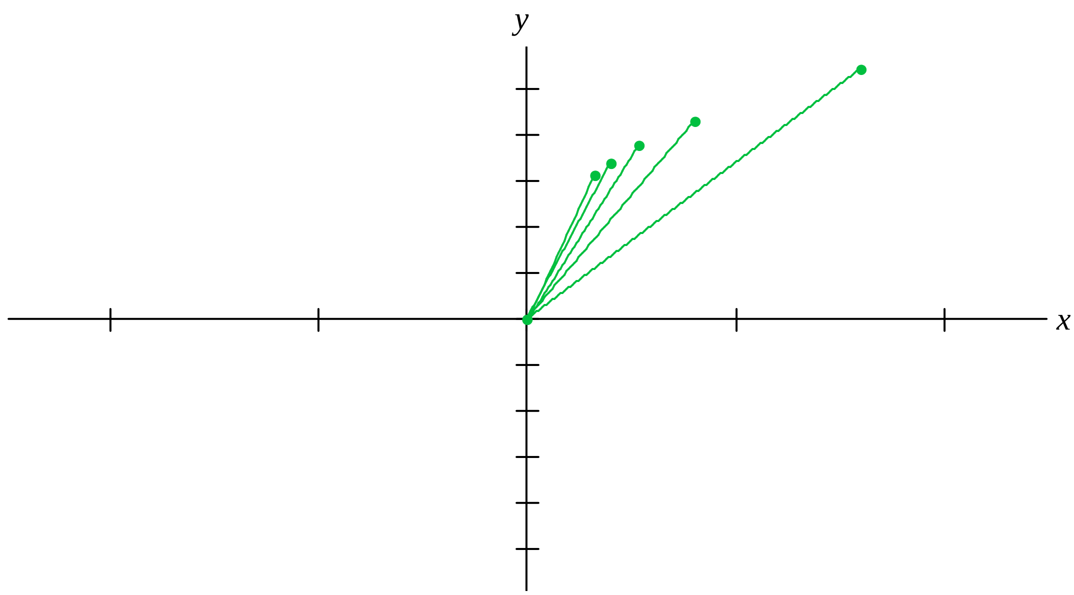





y
$y = \pm x^2$
0.02
0.01
-0.2
-0.1
0.1
0.2
x
-0.01
-0.02


y
0.02
0.01
-0.2
-0.1
0.1
0.2
x
-0.01
-0.02

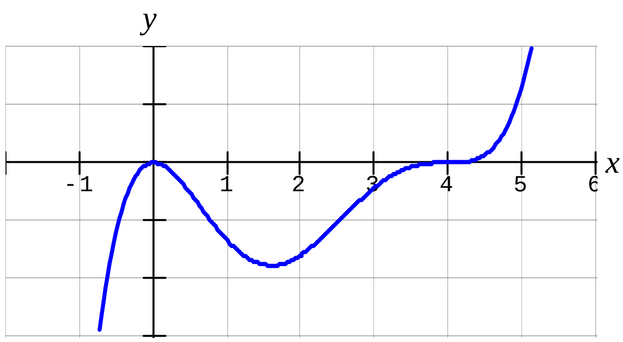

y
0.5
-0.5
x
-0.2
-0.1
0.1
0.2
$y = \pm\sqrt[3]{|x|}$


y
x

3. For what values of k will $m(x) = \begin{cases} x^k \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ be differentiable at $x = 0$? Make a conjecture based on the cases above, and then prove that your conjecture is correct.

Calculus: First and Second Derivative Name ___________________________

1. Use the graph provided to find any values of x at which the function $y = f(x)$ has the indicated feature in each of the following situations. If there is no such value, write "None". If the information *cannot be determined*, write CBD.



(a) Suppose the graph shown is the graph of $f(x)$. Find the following for $f(x)$:

Critical points: __________ Local maxima: __________ Local minima: ____________

Global maxima: _________ Global minima: _________ Inflection points: __________

(b) Suppose the graph shown is the graph of $f'(x)$, the *first* derivative of f. Find the following for $f(x)$:

Critical points: __________ Local maxima: __________ Local minima: ____________

Global maxima: _________ Global minima: _________ Inflection points: __________

(c) Suppose the graph shown is the graph of $f''(x)$, the *second* derivative of f. Find the following for $f(x)$:

Critical points: __________ Local maxima: __________ Local minima: ____________

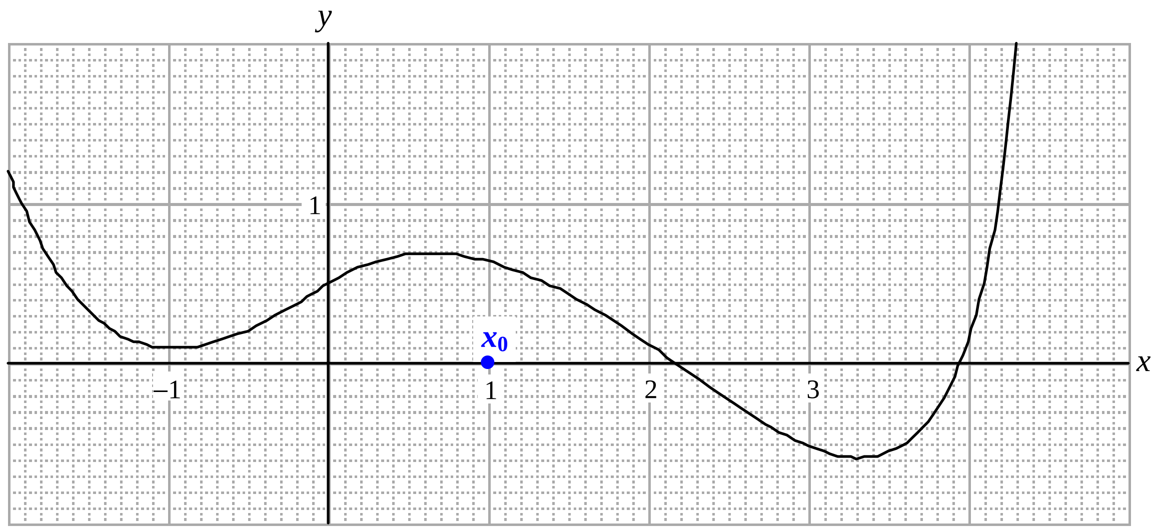
Global maxima: _________ Global minima: _________ Inflection points: __________

2. In debating the dependence of the U.S. on foreign oil supplies, two opposing sides made the following arguments. Assume that consumption of foreign oil is a function of time, $C = f(t)$. Which derivative of f does each statement refer to, and what would be the sign of that derivative according to the statement?

	Derivative	Sign
"The consumption of foreign oil keeps rising each year!"		
"Yes, but the increase in oil consumption is less each year!"		

Calculus: Newton's Method Name ___________________________

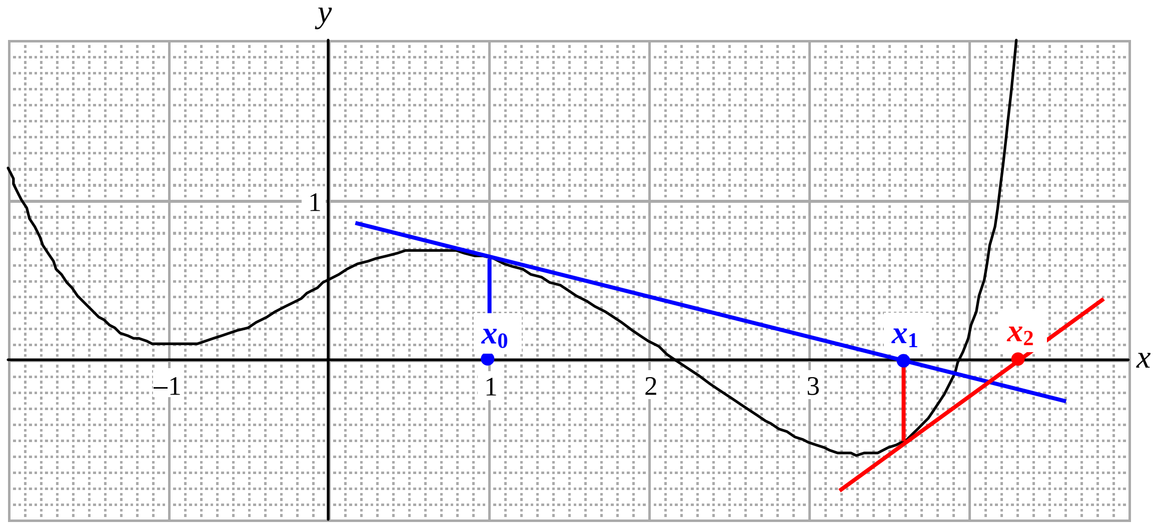
To find the value of the root close to $x = 2$, Newton's method is used, with a starting value $x_0 = 1$. (Each small square represents 0.1 unit.)

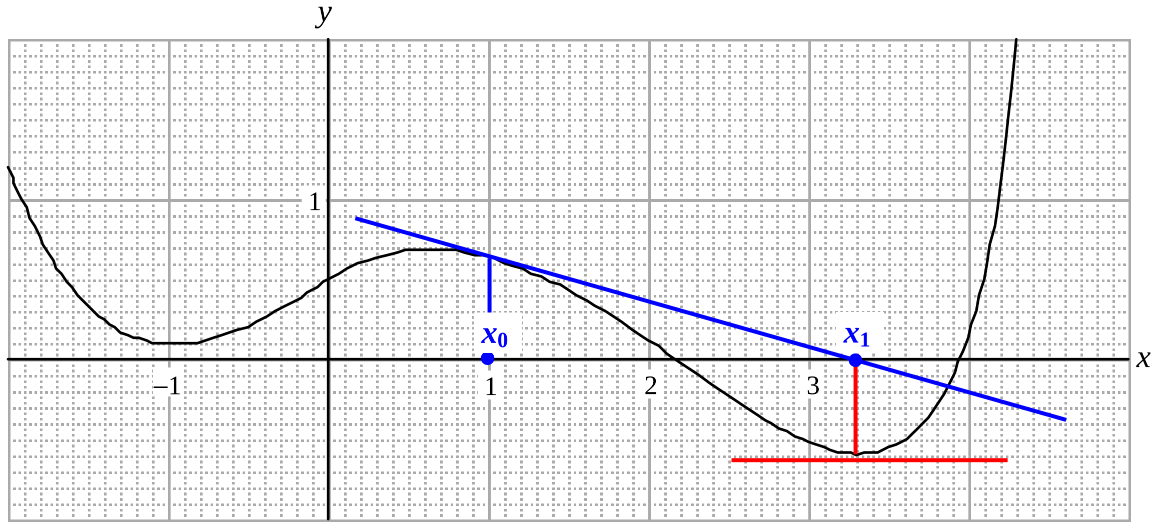


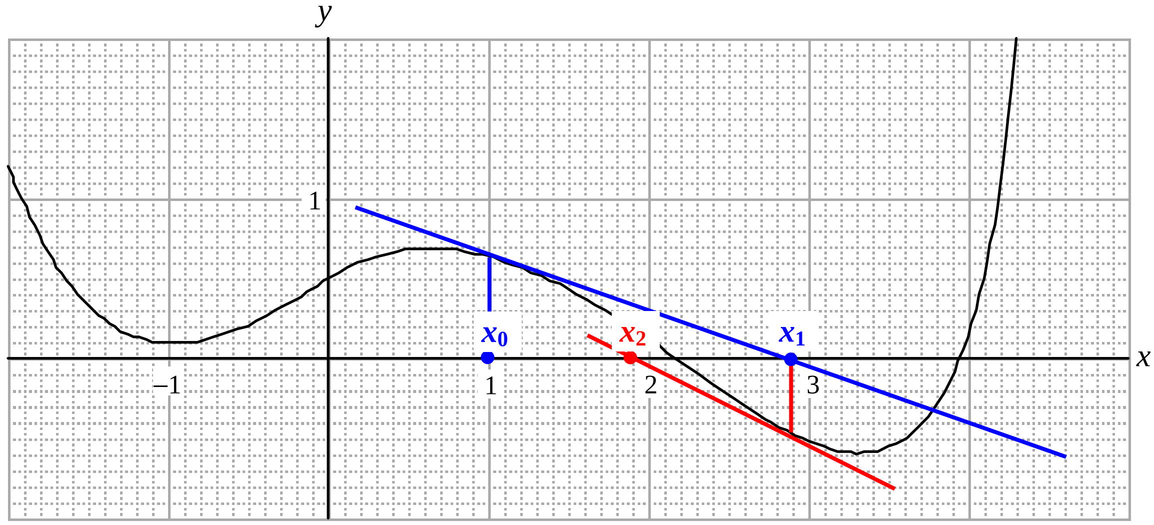
1. Locate the next two approximations, x_1 and x_2, on the graph. Show by manner of a sketch how these values were determined.

2. If we continue to find additional approximations, what is likely to happen? Explain in a sentence or two.

Calculus: Newton's Method



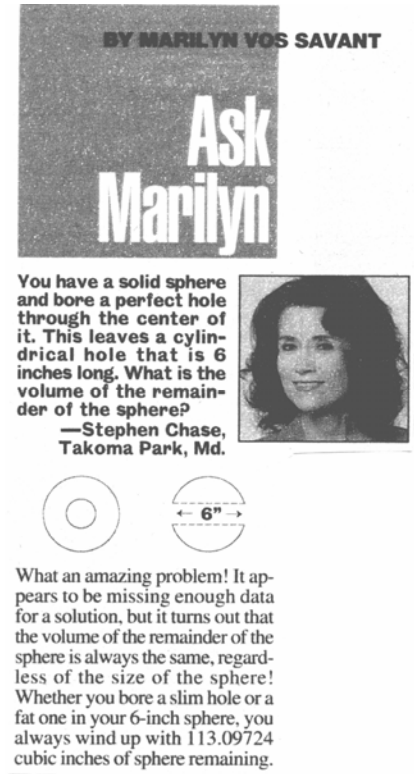




Calculus: Volumes of Revolution Name ___________________________

The article shown at right appeared in Parade magazine. The columnist provides an approximate solution to the stated problem. Is she right? You are to complete the following two tasks.

(a) Using six inches as the length of the cylindrical hole, find the exact form of the solution to the stated problem. Comment on the accuracy of Marilyn's answer. (Note that six inches is the length of the side of the hole, NOT the diameter of the sphere. The diameter of the sphere will change to accommodate the width of the cylindrical hole. That is, wider holes will require larger spheres.)



BY MARILYN VOS SAVANT

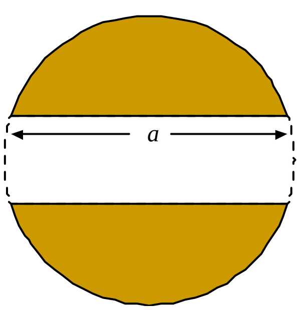
Ask Marilyn

You have a solid sphere and bore a perfect hole through the center of it. This leaves a cylindrical hole that is 6 inches long. What is the volume of the remainder of the sphere?
—Stephen Chase, Takoma Park, Md.

← 6" →

What an amazing problem! It appears to be missing enough data for a solution, but it turns out that the volume of the remainder of the sphere is always the same, regardless of the size of the sphere! Whether you bore a slim hole or a fat one in your 6-inch sphere, you always wind up with 113.09724 cubic inches of sphere remaining.

(b) Generalize this problem! Let the length of the cylindrical hole be *a* inches. Find the remaining volume in terms of *a* in exact form.



Calculus: Multivariable Limits Name ___________________________

1. Investigate $f(x,y)=\begin{cases} \dfrac{(x+y)(\sin x+\sin y)}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 1 & \text{if } (x,y)=(0,0) \end{cases}$ as (x, y) approaches (0, 0).

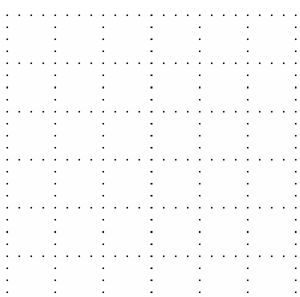
(a) *Numerically:* Complete each table of values. In the first table, (x, y) is approaching (0, 0) along the x-axis, and in the second table it is approaching along the y-axis.

x	*y*	*f*(*x*, *y*)
0.5	0	
0.1	0	
0.01	0	
–0.5	0	
–0.1	0	
–0.01	0	

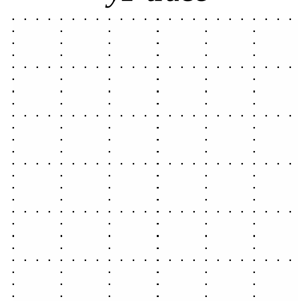
x	*y*	*f*(*x*, *y*)
0	0.5	
0	0.1	
0	0.01	
0	–0.5	
0	–0.1	
0	–0.01	

(b) *Graphically:* Sketch a graph of the xz-trace ($z = f(x, 0)$) and the yz-trace ($z = f(0, y)$).

xz-trace



yz-trace



(c) *Algebraically:* Find each limit (the first limit is along the x-axis, and the second is along the y-axis).

$\lim\limits_{(x,0)\to(0,0)} f(x,y) =$

$\lim\limits_{(0,y)\to(0,0)} f(x,y) =$

Based on these results, what does the limit appear to be as (x, y) approaches (0, 0)?

Does $f(x, y)$ appear to be continuous at (0, 0)? Explain why or why not.

2. Continue to investigate the function $f(x, y)$ as (x, y) approaches (0, 0).

(a) *Numerically:* Complete each table of values. In the first table, (x, y) is approaching (0, 0) along the line $y = x$, and in the second table it is approaching along the line $y = -x$.

x	***y***	***f*(*x*, *y*)**
0.5	0.5	
0.1	0.1	
0.01	0.01	

x	***y***	***f*(*x*, *y*)**
0.5	–0.5	
0.1	–0.1	
0.01	–0.01	

(b) *Graphically:* Sketch a graph of the cross-section along the line $y = x$ ($z = f(x, x)$) and the cross-section along the line $y = -x$ ($z = f(x, -x)$).

along $y = x$

along $y = -x$

(c) *Algebraically:* Find each limit (the first limit is along the line $y = x$, and the second is along the line $y = -x$).

$\lim_{(x,x)\to(0,0)} f(x, y) =$

$\lim_{(x,-x)\to(0,0)} f(x, y) =$

Based on these latest results, what can be said about the limit as (x, y) approaches (0, 0)?

Is $f(x, y)$ continuous at (0, 0)? Explain why or why not.